JET-FLOW PARAMETERS OF A SPECIAL FORM OF IDEAL HEAVY FLUID

Yu. I. Petukhov

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In [2] the possibility is indicated of effectively using the approximation [1] of the boundary condition on the free surface of an ideal heavy fluid for the study of jet flows with polygonal solid boundaries, whose separate sections are inclined to the horizon at angles $\pi/2$ and $\pi/6$. In this study the analytic function Φ is introduced, whose imaginary part becomes zero on the free boundary.

The present study concerns the jet flow of an ideal heavy fluid with a free boundary in a geometry such that either the real or the imaginary parts of function Φ are equal to zero on every flow boundary. This allows us to establish the function with accuracy up to several constants and then obtain the values of the constants from an analysis of additional information about the problem.

Let there be the steady plane flow of an ideal heavy fluid which is represented in Fig. 1. Here AD, DC, and AE are solid walls inclined to the x axis at the angles indicated in the figure. The fluid flows uniformly from an infinitely distant point A and descends along a tangent to AE at point B; the jet with the free boundary BC flows along the infinite wall DC. A scheme of nondetached flow around angle D is assumed, which indicates that the velocity becomes infinite at this point. The fluid-flow rate Q and the channel width H at point A are known. The acceleration of free fall g is oriented in the negative direction of the y axis. We must determine the form of the free boundary and the coordinates of the descent point B.

We assume parameter H for the linear scale, quantity Q for the scale of the potential and the stream function, and ratio Q/H for the scale of the velocity. We introduce the dimensionless variable of the physical plane $z = x + iy$ and the dimensionless complex potential $W = \varphi + i\psi$. In the z plane the channel width at point A is equal to unity, and in the W plane the zone of flow is a strip of unit width. By freely choosing the potential level, we assume that the potential is equal to zero at point D. We introduce the auxiliary complex variable t so that the upper half-plane corresponds to the zone of flow of the t plane. The variables W and t together with the correspondence of the points indicated in Fig. 2 are related by the equation.

$$W = \frac{1}{\pi} \ln (t + 1) . \tag{1}$$

Since the potential of point B is unknown, we introduce parameter μ which characterizes the location of the point on the real axis of the t plane.

It is known [3] that the condition of pressure constancy on the free boundary can be written in the following differential form:

$$\frac{\nu}{3} \frac{dv^3}{d\varphi} + \sin\theta = 0, \qquad \nu = \frac{Q^2}{H^3 g} . \tag{2}$$

Here θ is the angle, which is measured counterclockwise from the x axis, between the fluid velocity vector on the free boundary and the x axis. We carry out the further transformation of condition (2) by following [2]. We introduce the function τ by the equation $v = e^{\tau}$ and we assume angle θ to be small, so that $\sin\theta \approx {}^1/_3 \sin 3\theta$.

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Let $\xi = -\tau + i\theta$

$$\Phi = -iv\frac{d\xi}{dW} + \frac{1}{3}e^{3\xi} \tag{3}$$

$$\mathrm{Re}\,\Phi = v\frac{\partial\theta}{\partial\varphi} + \frac{1}{3}e^{-3\tau}\cos 3\theta$$

$$\mathrm{Im}\,\Phi = v\frac{\partial\tau}{\partial\varphi} + \frac{1}{3}e^{-3\tau}\sin 3\theta\ .$$

We can establish that the condition on the free boundary is formulated in new variables as $\mathrm{Im}\Phi = 0$.

From (3) for the real part of function Φ we can see that on the rectangular-solid flow boundaries which form angles $\theta = \pm\pi/6$ with the x axis, $\mathrm{Re}\,\Phi = 0$. Thus, on every real axis of the t plane, we have

$$\begin{array}{ll} \mathrm{Im}\,\Phi = 0 & (-\infty, -\mu) \\ \mathrm{Re}\,\Phi = 0 & (-\mu, -1) \\ \mathrm{Re}\,\Phi = 0 & (-1, 0) \\ \mathrm{Re}\,\Phi = 0 & (0, \infty)\ . \end{array} \tag{4}$$

The function that satisfies (4) can be established by means of the Keldysh–Sedov formula [4] if the location and order of its poles are known. We study the singularities of function Φ in the case under study, for which we introduce the new function $\chi = e^{-3\xi}$ and rewrite (3) as

$$\Phi = \frac{iv\pi}{3}(t+1)\frac{1}{\chi}\frac{d\chi}{dt} + \frac{1}{3}\frac{1}{\chi}\ . \tag{5}$$

From (3) it follows that for $t = -1$ function Φ does not possess singularity. We examine the behavior of the function for $t \to +\infty$. Since the large real values of the argument correspond to points on DC of the physical plane, then $\mathrm{Re}\,\Phi = 0$. We investigate the behavior of $\partial\tau/\partial\varphi$. Since $\tau = \ln v$, and from the Bernoulli integral $v \sim \sqrt{A-y}$, where A is a certain constant, then

$$\frac{\partial\tau}{\partial\varphi} = \frac{1}{v}\frac{\partial v}{\partial\varphi} \sim \frac{1}{v^2}\frac{\partial v}{\partial y}\ .$$

This expression approaches zero for $y \to -\infty (t \to +\infty)$. Consequently, $\Phi \to 0$. The quantity Φ behaves the same for $t \to -\infty$. Point B is also a regular point. At point D the velocity becomes infinite (function χ has a pole of the first order for $t = 0$). From (5) it follows that, independent of the order of the pole of function χ (important as it is), function Φ has a pole of the first order for $t = 0$. At the remaining points of the flow it is analytic.

The information obtained allows us by means of the Keldysh–Sedov formula [4] to establish

$$\Phi = c + ia\sqrt{\mu + t}\,/\,t\ . \tag{6}$$

Here c and a are constants to be determined. Since for $t \to \infty$ $\Phi \to 0$, then $c = 0$.

We examine the expressions for the real and imaginary parts of function Φ for $t \to -1$. We have $\mathrm{Re}\Phi \to 0$.

The term $\nu(\partial\tau/\partial\varphi) \to 0$ for $t \to -1$. Thus, $\lim_{t\to -1}\Phi = i/3$. Hence and from relation (6) we obtain $t \to -1$

$$a\sqrt{\mu - 1} = -1/3\ . \tag{7}$$

For determining the parameters in (6) we must still construct another equation. For this we rewrite (5) in the form

$$(t+1)\frac{d\chi}{dt} + \frac{3i}{\pi v}\Phi\chi = \frac{i}{\pi v}\ . \tag{8}$$

Equation (8) together with the known function Φ is the differential equation for finding χ.

The singularity of function Φ at point $t = 0$ determines the form of the singularity of function χ at this point. Since it follows from physical considerations that χ should have a pole of the first order at this point, we can determine what Φ is for $t \to 0$.

Let, in the vicinity of $t = 0$, $\chi \approx B/t$, where B is a constant.

Thus, from (8) for $t \to 0$ it follows that

$$-B/t^2 + \frac{3i}{\pi v}\frac{ia\sqrt{\mu}}{t}\frac{B}{t} = 0, \qquad \frac{3a\sqrt{\mu}}{\pi v} = -1\ . \tag{9}$$

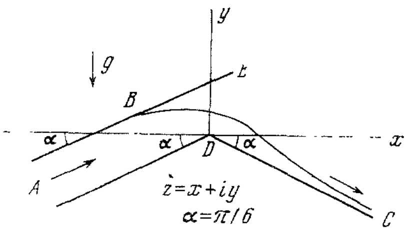

Fig. 1

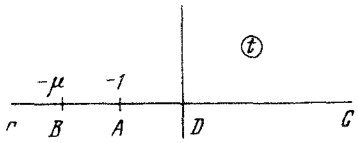

Fig. 2

From (8) and (9) we obtain

$$\mu = \pi^2\nu^2 / (\pi^2\nu^2 - 1), \qquad a = \sqrt{\pi^2\nu^2 - 1}/3 \ . \tag{10}$$

Function Φ is completely determined by Eqs. (6) and (10) (c = 0).

The first equation of (10) determines by means of Eq. (1) the flow potential at the descent point.

The problem is completely solved, since function χ is found from Eq. (8). The constant appearing in the integration of the equation is found from the Bernoulli integral which is written for any two values of t on the free boundary. The transformation from variable t to the physical variable z is realized according to the equation [3]

$$z = \int_0^t \frac{dW}{dt} \frac{e^{i\theta} dt}{v} \ .$$

Here v is the absolute value of the velocity and θ is the angle between the x axis and the velocity vector.

We can see from the determination of parameter μ that it is positive. It follows from (10) that flow with tangential descent at a point (Fig. 1) is possible only for the condition $\pi\nu > 1$.

For small values of ν, we can expect that the free boundary in the neighborhood of point B will be horizontal. The streamline at this point will undergo a break (the angle of break is 5π/6, and in the previous case, π). By an analysis similar to the one carried out, we can establish that Φ in this case takes the form

$$\Phi = ia / t \sqrt{\mu - t} \ . \tag{11}$$

From (11) and from the established properties of Φ for the parameters a and μ, we have

$$\mu = 1 / (1 - \pi^2\nu^2), \qquad a = -\pi\nu / 3 \sqrt{1 - \pi^2\nu^2} \, . \tag{12}$$

The parameter μ is also positive in this case. Consequently, a flow of such type is possible only for $\pi\nu < 1$.

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